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Estimates for Hilbert transforms along variable general curves

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ABSTRACT

We consider the problem of proving $L^2(\mathbb{R}^2)$ boundedness and single annulus $L^p(\mathbb{R}^2)$ estimate for the Hilbert transform along variable general curve $(t, u_1(x_1)t + u_2(x_1)\gamma(t))$

$$H_{u_1,u_2,\gamma}f(x_1,x_2) := \mathrm{p.\,v.}\int_{-\infty}^{\infty} f(x_1-t, x_2-u_1(x_1)t-u_2(x_1)\gamma(t))\,\frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2)\in\mathbb{R}^2,$$

where $p \in (1,\infty)$, γ is a general curve on $\mathbb{R}$, $u_1 : \mathbb{R} \to \mathbb{R}$ and $u_2 : \mathbb{R} \to \mathbb{R}$ are measurable functions. Moreover, all the bounds are independent of the measurable functions u_1 and u_2. For any given $p \in (1,\infty)$, we also obtain the $L^p(\mathbb{R})$ boundedness of the corresponding Carleson operator

$$\mathcal{C}_{N_1,N_2,\gamma}f(x) := \sup_{N_1,N_2\in\mathbb{R}}\left|\mathrm{p.\,v.}\int_{-\infty}^{\infty} e^{iN_1t+iN_2\gamma(t)}f(x-t)\,\frac{\mathrm{d}t}{t}\right|, \quad \forall\,x\in\mathbb{R}.$$

1. Introduction

For any two given measurable functions $u_1 : \mathbb{R} \to \mathbb{R}$, $u_2 : \mathbb{R} \to \mathbb{R}$ and a general curve γ, the *Hilbert transform* $H_{u_1,u_2,\gamma}$ along variable general curve $(t, u_1(x_1)t + u_2(x_1)\gamma(t))$ is defined by

$$H_{u_1,u_2,\gamma}f(x_1,x_2) := \mathrm{p.\,v.}\int_{-\infty}^{\infty} f(x_1-t, x_2-u_1(x_1)t-u_2(x_1)\gamma(t))\,\frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2)\in\mathbb{R}^2, \tag{1.1}$$

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for any function f in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, here and hereafter, $\mathrm{p.v.}\int_{-\infty}^{\infty}$ denotes the principal-value integral. The corresponding *Carleson operator* $\mathcal{C}_{N_1,N_2,\gamma}$ is defined by

$$\mathcal{C}_{N_1,N_2,\gamma}f(x) := \sup_{N_1,N_2\in\mathbb{R}}\left|\mathrm{p.v.}\int_{-\infty}^{\infty} e^{iN_1t+iN_2\gamma(t)}f(x-t)\,\frac{\mathrm{d}t}{t}\right|,\quad \forall\, x\in\mathbb{R}, \tag{1.2}$$

for any $f\in\mathcal{S}(\mathbb{R})$. By linearizing the supremum, the estimate $\|\mathcal{C}_{N_1,N_2,\gamma}f\|_{L^p(\mathbb{R})}\le C\|f\|_{L^p(\mathbb{R})}$ is equivalent to obtaining $\|\mathcal{C}_{u_1,u_2,\gamma}f\|_{L^p(\mathbb{R})}\le C\|f\|_{L^p(\mathbb{R})}$ for the *Carleson operator* $\mathcal{C}_{u_1,u_2,\gamma}$ defined as

$$\mathcal{C}_{u_1,u_2,\gamma}f(x) := \mathrm{p.v.}\int_{-\infty}^{\infty} e^{iu_1(x)t+iu_2(x)\gamma(t)}f(x-t)\,\frac{\mathrm{d}t}{t},\quad \forall\, x\in\mathbb{R}, \tag{1.3}$$

for any $f\in\mathcal{S}(\mathbb{R})$, where $u_1:\ \mathbb{R}\to\mathbb{R}$ and $u_2:\ \mathbb{R}\to\mathbb{R}$ are measurable functions, and the bound C must be independent of the measurable functions u_1 and u_2. In this paper, we want to establish the $L^2(\mathbb{R}^2)$ boundedness and single annulus $L^p(\mathbb{R}^2)$ estimate for the Hilbert transform $H_{u_1,u_2,\gamma}$ and the $L^p(\mathbb{R})$ boundedness of the Carleson operator $\mathcal{C}_{u_1,u_2,\gamma}$ for some general plane curves γ, where $p\in(1,\infty)$.

One motivation of this paper is the longstanding Stein conjecture. Let $v:\ \mathbb{R}^2\to\mathbb{S}^1$ be a measurable unit circle and for any $f\in\mathcal{S}(\mathbb{R}^2)$, define

$$H_{v,\varepsilon}f(x) := \mathrm{p.v.}\int_{-\varepsilon}^{\varepsilon} f(x-v(x)t)\,\frac{\mathrm{d}t}{t},\quad \forall\, x\in\mathbb{R}^2. \tag{1.4}$$

Stein [29] conjectured that $H_{v,\varepsilon}$ is weak-$(2,2)$ bounded whenever v is a Lipschitz function with $\|v\|_{\mathrm{Lip}}\approx\epsilon^{-1}$. For smooth vector field v, Christ et al. [10] obtained the $L^p(\mathbb{R}^2)$ boundedness of $H_{v,\varepsilon}$ for all $p\in(1,\infty)$ under some extra curvature conditions. Later, Lacey and Li [21] established the $L^2(\mathbb{R}^2)$ boundedness of $H_{v,\varepsilon}$ if $v\in C^{1+\alpha}$ with $\alpha>0$ and a suitable Kakeya maximal operator is bounded. On the other hand, the *Hilbert transform* H_v along the vector field v without cut-off is defined by

$$H_vf(x) := \mathrm{p.v.}\int_{-\infty}^{\infty} f(x-v(x)t)\,\frac{\mathrm{d}t}{t},\quad \forall\, x\in\mathbb{R}^2, \tag{1.5}$$

for any $f\in\mathcal{S}(\mathbb{R}^2)$. Lacey and Li [20] obtained single annulus $L^p(\mathbb{R}^2)$ estimate for H_v. They showed that H_vP_k maps $L^2(\mathbb{R}^2)$ into $L^{2,\infty}(\mathbb{R}^2)$ and is bounded on $L^p(\mathbb{R}^2)$ with $p\in(2,\infty)$ for an arbitrary measurable vector field v and these operators' norms are independent of $k\in\mathbb{Z}$. Here P_k denotes the k-th Littlewood-Paley projection operator.

Since H_v becomes Hilbert transform when v equals to $(0,1)$ or $(0,-1)$, we should only consider the vector fields with non-vanishing first component. By scaling, for any $f\in\mathcal{S}(\mathbb{R}^2)$, we can assume that $v(x_1,x_2):=(1,U(x_1,x_2))$ and consider

$$H_Uf(x_1,x_2) := \mathrm{p.v.}\int_{-\infty}^{\infty} f\left(x_1-t,x_2-U(x_1,x_2)t\right)\,\frac{\mathrm{d}t}{t},\quad \forall\,(x_1,x_2)\in\mathbb{R}^2,$$

where $U:\ \mathbb{R}^2\to\mathbb{R}$ is a measurable function. It is a special case of the following *Hilbert transform* $H_{U,\gamma}$ along variable plane curve $(t,U(x_1,x_2)\gamma(t))$:

$$H_{U,\gamma}f(x_1,x_2) := \mathrm{p.v.}\int_{-\infty}^{\infty} f(x_1-t, x_2-U(x_1,x_2)\gamma(t))\,\frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2)\in\mathbb{R}^2, \tag{1.6}$$

original defined for any $f\in\mathcal{S}(\mathbb{R}^2)$. There are some results about the Hilbert transform $H_{U,\gamma}$ in (1.6); see, for example, [13,15,27]. The boundedness of the Hilbert transform $H_{U,\gamma}$ for general plane curve γ is still open.

We know that $H_{U,\gamma}$ is not bounded on $L^p(\mathbb{R}^2)$ for all $p\in(1,\infty)$, if we only assume U is a measurable function. The assumption that Lipschitz regularity of U plays a crucial role in the $L^p(\mathbb{R}^2)$ boundedness of $H_{U,\gamma}$ for any given $p\in(1,\infty)$, even one cannot take U to be Hölder continuous of any index strictly less than one; see [16,21]. But if we consider that the measurable function U depends only on one variable, i.e. $U(x_1,x_2)=u(x_1)$ for any $(x_1,x_2)\in\mathbb{R}^2$, the Lipschitz regularity of U in $H_{U,\gamma}$ can be ignored. In fact, the first author and Li [24] have established the $L^p(\mathbb{R}^2)$ boundedness of the *Hilbert transform* $H_{u,\gamma}$ along variable plane curve $(t,u(x_1)\gamma(t))$, which is defined by setting, for any $f\in\mathcal{S}(\mathbb{R}^2)$,

$$H_{u,\gamma}f(x_1,x_2) := \mathrm{p.v.}\int_{-\infty}^{\infty} f(x_1-t, x_2-u(x_1)\gamma(t))\,\frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2)\in\mathbb{R}^2, \tag{1.7}$$

for any given $p\in(1,\infty)$ with the bound independent of the measurable function u.

We now introduce another *Hilbert transform* H_u, which is defined as following, for any $f\in\mathcal{S}(\mathbb{R}^2)$,

$$H_u f(x_1,x_2) := \mathrm{p.v.}\int_{-\infty}^{\infty} f(x_1-t, x_2-u(x_1)t)\,\frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2)\in\mathbb{R}^2, \tag{1.8}$$

where $u:\ \mathbb{R}\to\mathbb{R}$ is a measurable function. Continuing the work in [24], we combine the Hilbert transform $H_{u,\gamma}$ in (1.7) with H_u in (1.8) and consider

$$H_{u_1,u_2,\gamma}f(x_1,x_2) := \mathrm{p.v.}\int_{-\infty}^{\infty} f(x_1-t, x_2-u_1(x_1)t-u_2(x_1)\gamma(t))\,\frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2)\in\mathbb{R}^2,$$

original defined for any $f\in\mathcal{S}(\mathbb{R}^2)$, where $u_1:\ \mathbb{R}\to\mathbb{R}$ and $u_2:\ \mathbb{R}\to\mathbb{R}$ are measurable functions. This is the operator (1.1), which has been introduced in [16] with the restriction that γ is $|t|^\alpha$ or $\mathrm{sgn}(t)|t|^\alpha$ for any $t\in\mathbb{R}$, $\alpha\in(0,\infty)$, $\alpha\neq1$, $\alpha\neq2$. When $u_1=u_2$, the Hilbert transform $H_{u_1,u_2,\gamma}$ is demoted to the Hilbert transform $H_{u,\gamma}$ along a new variable plane curve $(t,u(x_1)(t+\gamma(t)))$. A fundamental question concerns suitable conditions on the curve γ, under which $H_{u_1,u_2,\gamma}$ is bounded on $L^p(\mathbb{R}^2)$ for some $p\in(1,\infty)$. In this paper, we provide a sufficient condition of curve γ to this question.

We now state one of our main results. For the Hilbert transform $H_{u_1,u_2,\gamma}$, we have

Theorem 1.1. *Let $u_1:\ \mathbb{R}\to\mathbb{R}$ and $u_2:\ \mathbb{R}\to\mathbb{R}$ be measurable functions, $\gamma\in C^4(\mathbb{R})$ be either odd or even, with $\gamma(0)=\gamma'(0)=0$, and convex on $(0,\infty)$, satisfying*

(i) $\frac{\gamma'(2t)}{\gamma'(t)}$ *is decreasing and bounded by a constant C_1 from above on $(0,\infty)$,*
(ii) *there exists a positive constant C_2 such that $\frac{t\gamma''(t)}{\gamma'(t)}\geq C_2$ on $(0,\infty)$,*
(iii) *there exists a positive constant C_3 such that $|\frac{t\gamma'''(t)}{\gamma''(t)}|\leq C_3$ on $(0,\infty)$,*
(iv) *there exists a positive constant C_4 such that $|(\frac{\gamma'''}{\gamma''})'(t)|\geq\frac{C_4}{t^2}$ on $(0,\infty)$,*
(v) $\frac{\gamma^{(4)}(t)}{\gamma'''(t)}$ *is strictly monotone or equals to a constant on $(0,\infty)$.*

Then there exists a positive constant C such that

$$\|H_{u_1,u_2,\gamma}f\|_{L^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}$$

for any $f \in L^2(\mathbb{R}^2)$ *with the bound* C *independent of* u_1 *and* u_2*.*

Throughout this paper, we always use C to denote a *positive constant*, independent of the main parameters involved, but whose value may change at each occurrence. The *positive constants with subscripts*, such as C_1 and C_2, do not change in different occurrences. For two real functions f and g, we always use $f \lesssim g$ or $g \gtrsim f$ to denote that f is smaller than a positive constant C times g, and we always use $f \approx g$ as shorthand for $f \lesssim g \lesssim f$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function supported on $\{t \in \mathbb{R} : \frac{1}{2} \le |t| \le 2\}$ such that $\Sigma_{l\in\mathbb{Z}}\psi_l(t) = 1$ for any $t \neq 0$, where $\psi_l(t) := \psi(2^{-l}t)$. For any $l \in \mathbb{Z}$, let P_l denote the Littlewood-Paley projection operator in the second variable associated with ψ_l by

$$P_l f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - z)\check{\psi}_l(z)\,\mathrm{d}z,$$

where $\check{\psi}_l$ means the Fourier inverse transform of ψ_l. For the single annulus $L^p(\mathbb{R}^2)$ estimate for the Hilbert transform $H_{u_1,u_2,\gamma}$, we have the following

Theorem 1.2. *Let* u_1*,* u_2 *and* γ *be the same as in Theorem 1.1. Then for any given* $p \in (1, \infty)$*, there exists a positive constant* C *such that*

$$\|H_{u_1,u_2,\gamma}P_l f\|_{L^p(\mathbb{R}^2)} \le C\|P_l f\|_{L^p(\mathbb{R}^2)}$$

uniformly in $l \in \mathbb{Z}$ *with the bound* C *independent of* u_1 *and* u_2*.*

Remark 1.3. Let γ be $|t|^\alpha$ or $\mathrm{sgn}(t)|t|^\alpha$ for any $t \in \mathbb{R}$, $\alpha \in (0, \infty)$, $\alpha \neq 1$, $\alpha \neq 2$, Guo et al. [16, Theorem 1.6] obtained the single annulus $L^p(\mathbb{R}^2)$ estimate for $H_{u_1,u_2,\gamma}$ for any given $p \in (1, \infty)$. Therefore, Theorem 1.2 can be viewed as an extension of Guo et al. [16, Theorem 1.6] from the homogeneous curve $|t|^\alpha$ or $\mathrm{sgn}(t)|t|^\alpha$ to a more general curve. It is easy to see that γ satisfies $\gamma(ab) = \gamma(a)\gamma(b)$ for all $a, b \in (0, \infty)$ if γ is the homogeneous curve $|t|^\alpha$ or $\mathrm{sgn}(t)|t|^\alpha$, which plays a crucial role in Guo et al.'s proof. But this property can not be hold by a general curve γ, this is the main difficulty we have overcome, thus our results make contribution in this direction.

Remark 1.4. The condition of Theorem 1.1(v) is used to obtain Lemma 2.3, which plays a crucial role in the proof of Proposition 2.1. From the proof of Lemma 2.3, the condition of Theorem 1.1(v) can be replaced by a weaker condition: for any $c \neq 0$, the equation $\frac{\gamma^{(4)}(t-c)}{\gamma'''(t-c)} = \frac{\gamma^{(4)}(t)}{\gamma'''(t)}$ on $t \in \mathbb{R}$ and $t \neq 0$, $t \neq c$, has a finite number of solutions including no solution, or there are at most a finite number of intervals such that the equation above is established on each of the intervals, or both, where the number is independent of c.

Remark 1.5. In [24], the authors obtained the $L^p(\mathbb{R}^2)$ boundedness of the Hilbert transform $H_{u,\gamma}$ in (1.7) for any given $p \in (1, \infty)$. For $H_{u_1,u_2,\gamma}$, this method used in [24] is no longer valid, one of the difficulties is the open problem that whether the maximal operator

$$\mathcal{M}_u f(x_1, x_2) := \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - t, x_2 - u(x_1)t)|\,\mathrm{d}t, \quad \forall\,(x_1, x_2) \in \mathbb{R}^2,$$

is bounded on $L^p(\mathbb{R}^2)$ for some $p \in (1, \infty)$.

Remark 1.6. There are some curves γ satisfying all the conditions of Theorem 1.1. We here only write the part $t \in [0, \infty)$, and define $\gamma(t) := \pm\gamma(-t)$ for $t \in (-\infty, 0]$. For example,

(i) for any $t \in [0, \infty)$, $\gamma(t) := t^\alpha, \alpha \in (1, \infty), \alpha \neq 2$,
(ii) for any $t \in [0, \infty)$, $\gamma(t) := \int_0^t \tau^2 \log(1 + \tau)\, \mathrm{d}\tau$.

Another motivation of the above study is the *Hilbert transform* H_γ along curve $(t, \gamma(t))$:

$$H_\gamma f(x_1, x_2) := \mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \gamma(t)) \frac{\mathrm{d}t}{t}, \quad \forall\, (x_1, x_2) \in \mathbb{R}^2, \tag{1.9}$$

which has been studied extensively; see, for example, [4,5,8,31,32]. If $u_1 : \mathbb{R} \to \mathbb{R}$ equals to 0 and $u_2 : \mathbb{R} \to \mathbb{R}$ is a constant λ, $\lambda \in \mathbb{R}$, then the operator in (1.1) becomes the following *directional Hilbert transform* $H_{\lambda,\gamma}$ along curve $(t, \gamma(t))$ defined for a fixed direction $(1, \lambda)$ as

$$H_{\lambda,\gamma} f(x_1, x_2) := \mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \lambda\gamma(t)) \frac{\mathrm{d}t}{t}, \quad \forall\, (x_1, x_2) \in \mathbb{R}^2,$$

whose $L^p(\mathbb{R}^2)$ boundedness can be obtained easily by the results about Hilbert transform H_γ. On the other hand, for all $p \in (1, \infty)$, it is not hard to get that

$$\sup_{\lambda \in \mathbb{R}} \|H_{\lambda,\gamma} f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

By linearizing the supremum, we get the $L^p(\mathbb{R}^2)$ boundedness of the corresponding maximal operator

$$\sup_{\lambda \in \mathbb{R}} |H_{\lambda,\gamma} f(x_1, x_2)|, \quad \forall\, (x_1, x_2) \in \mathbb{R}^2,$$

is equivalent to the $L^p(\mathbb{R}^2)$ boundedness of $H_{U,\gamma}$ under the assumption that U is a measurable function and the bound must be independent of the measurable function U. As we have already mentioned earlier, this is impossible. Therefore, we cannot hope to obtain an uniform constant C such that

$$\left\| \sup_{\lambda \in \mathbb{R}} |H_{\lambda,\gamma} f| \right\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$$

holds for all $f \in L^p(\mathbb{R}^2)$, where $p \in (1, \infty)$. Indeed, Guo et al. [18] have showed that

$$\left\| \sup_{\lambda \in U} |H_{\lambda,\gamma}| \right\|_{L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)} \approx \sqrt{\log(\mathfrak{N}(U))}$$

for all $p \in (2, \infty)$, where

$$\mathfrak{N}(U) := 1 + \sharp \left\{ n \in \mathbb{Z} : \ [2^n, 2^{n+1}] \cap U \neq \emptyset \right\}$$

and γ is homogeneous of degree b, with $\gamma(\pm 1) \neq 0$ and $b > 1$. Instead, from Theorem 1.1, we have that

$$\left\| \sup_{\lambda_1,\lambda_2\in\mathbb{R}^2} \left\| \mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \lambda_1 t - \lambda_2\gamma(t))\, \frac{\mathrm{d}t}{t} \right\|_{L^2(\mathbb{R}^1_{x_2})} \right\|_{L^2(\mathbb{R}^1_{x_1})} \le C\|f\|_{L^2(\mathbb{R}^2)},$$

which squeezes the supremum between the two L^2 norms on the left hand side.

For the Hilbert transform $H_{u_1,u_2,\gamma}$ in (1.1), when $u_2 := 0$, then the operator $H_{u_1,u_2,\gamma}$ is demoted to the Hilbert transform H_u in (1.8). Bateman [1] proved the single annulus $L^p(\mathbb{R}^2)$ estimate for H_u for any given $p \in (1,\infty)$. Later, Bateman and Thiele [2] obtained that H_u is bounded on $L^p(\mathbb{R}^2)$ for any given $p \in (\frac{3}{2},\infty)$. When $u_1 := 0$, then the operator $H_{u_1,u_2,\gamma}$ is demoted to the Hilbert transform $H_{u,\gamma}$ in (1.7). Let γ be $|t|^\alpha$ or $\mathrm{sgn}(t)|t|^\alpha$ for any $t \in \mathbb{R}$, $\alpha \in (0,\infty)$, $\alpha \neq 1$, Guo et al. [16] proved that $H_{u,\gamma}$ is bounded on $L^p(\mathbb{R}^2)$, where $p \in (1,\infty)$. Furthermore, Carbery et al. [6] obtained the $L^p(\mathbb{R}^2)$ boundedness of $H_{u,\gamma}$ if $u(x_1) := x_1$ for any $x_1 \in \mathbb{R}$, $\gamma \in C^3(\mathbb{R})$ is either odd or even, convex curve on $(0,\infty)$, and satisfies $\gamma(0) = \gamma'(0) = 0$ and the function $\frac{t\gamma''(t)}{\gamma'(t)}$ is decreasing and bounded below on $(0,\infty)$, where $p \in (1,\infty)$. Under the same assumptions, Bennett [3] established the $L^2(\mathbb{R}^2)$ boundedness of

$$H_{P,\gamma}f(x_1,x_2) := \mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - P(x_1)\gamma(t))\, \frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2) \in \mathbb{R}^2, \tag{1.10}$$

for any general polynomial P. More recently, Chen and Zhu [9] obtained the $L^2(\mathbb{R}^2)$ boundedness of $H_{P,\gamma}$ in (1.10) if the curvature condition for γ is replaced as $(\frac{\gamma''}{\gamma'})'(t) \le -\frac{\mu_1}{t^2}$ on $t \in (0,\infty)$ with $\mu_1 > 0$. The first author and Li [23] also obtained the $L^2(\mathbb{R}^2)$ boundedness of $H_{P,\gamma}$ in (1.10) by asking the curvature condition for $\gamma \in C^2(\mathbb{R})$ as

(i) $\frac{\gamma''(t)}{\gamma'(t)}$ is decreasing on $(0,\infty)$,
(ii) there exists a positive constant μ_2 such that $\frac{t\gamma''(t)}{\gamma'(t)} \ge \mu_2$ for any $t \in (0,\infty)$,
(iii) $\gamma''(t)$ is monotone on $(0,\infty)$.

These results are all based on iteration on the degree of polynomial P, which can not been applied to general measurable function. Thus, we must find a new method to dispose general measurable function, we refer the reader to [24] for this topic. Furthermore, the first author [33] obtained the $L^2(\mathbb{R}^2)$ boundedness and single annulus $L^p(\mathbb{R}^2)$ estimate for the *Hilbert transform* $H_{\alpha,\beta}$ along double variable fractional monomial $u_1(x_1)[t]^\alpha + u_2(x_1)[t]^\beta$, which is defined by setting, for any $f \in \mathcal{S}(\mathbb{R}^2)$,

$$H_{\alpha,\beta}f(x_1,x_2) := \mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha - u_2(x_1)[t]^\beta)\, \frac{\mathrm{d}t}{t}, \quad \forall\,(x_1,x_2) \in \mathbb{R}^2,$$

and the bounds are independent of the measurable functions u_1 and u_2, where $[t]^\alpha$ stands for either $|t|^\alpha$ or $\mathrm{sgn}(t)|t|^\alpha$, $[t]^\beta$ stands for either $|t|^\beta$ or $\mathrm{sgn}(t)|t|^\beta$ and $\alpha,\beta \in (1,\infty)$.

Now we turn to the Carleson operator $\mathcal{C}_{u_1,u_2,\gamma}$ in (1.3), which appears naturally in the study of the $L^2(\mathbb{R}^2)$ boundedness of the Hilbert transform $H_{u_1,u_2,\gamma}$. The original *Carleson operator* $\mathcal{C}$ is defined by

$$\mathcal{C}f(x) := \sup_{N\in\mathbb{R}} \left| \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iNt} f(x - t)\, \frac{\mathrm{d}t}{t} \right|, \quad \forall\, x \in \mathbb{R}, \tag{1.11}$$

for any $f \in \mathcal{S}(\mathbb{R})$. Carleson [7] proved that $\mathcal{C}$ is bounded on $L^2(\mathbb{R})$, which provided a critical step in obtaining almost everywhere convergence of Fourier series of $L^2(\mathbb{R})$ functions and also confirmed the outstanding Luzin conjecture. Hunt [19] obtained its $L^p(\mathbb{R})$ boundedness for any given $p \in (1, \infty)$. For more details about $\mathcal{C}$, we refer to [11,22,28,30].

Guo [14] considered a Carleson operator along a homogeneous curve $|t|^{\varepsilon_1}$ or $\operatorname{sgn}(t)|t|^{\varepsilon_2}$. He proved that, for any fixed $\varepsilon_1 \in \mathbb{R}$, $\varepsilon_1 \neq 1$ and $p \in (1, \infty)$, there exists a positive constant C_{p,ε_1} depending on p and ε_1 such that $\|\mathcal{C}^{\mathrm{even}}_{N,\varepsilon_1} f\|_{L^p(\mathbb{R})} \leq C_{p,\varepsilon_1}\|f\|_{L^p(\mathbb{R})}$, moreover, for any fixed $\varepsilon_2 \in \mathbb{R}$, $\varepsilon_2 \neq 0$ and $p \in (1, \infty)$, there exists a positive constant C_{p,ε_2} depending on p and ε_2 such that $\|\mathcal{C}^{\mathrm{odd}}_{N,\varepsilon_2} f\|_{L^p(\mathbb{R})} \leq C_{p,\varepsilon_2}\|f\|_{L^p(\mathbb{R})}$, where, for any $f \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{C}^{\mathrm{even}}_{N,\varepsilon_1} f(x) := \sup_{N\in\mathbb{R}} \left| \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iN|t|^{\varepsilon_1}} f(x-t)\,\frac{\mathrm{d}t}{t} \right|, \quad \forall\, x \in \mathbb{R},$$

and

$$\mathcal{C}^{\mathrm{odd}}_{N,\varepsilon_2} f(x) := \sup_{N\in\mathbb{R}} \left| \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iN\operatorname{sgn}(t)|t|^{\varepsilon_2}} f(x-t)\,\frac{\mathrm{d}t}{t} \right|, \quad \forall\, x \in \mathbb{R}.$$

Let $p \in (1, \infty)$, the first author and Li [24] obtained the $L^p(\mathbb{R})$ boundedness of the *Carleson operator* $\mathcal{C}_{N,\gamma}$ along a more general curve γ, which is defined by

$$\mathcal{C}_{N,\gamma} f(x) := \sup_{N\in\mathbb{R}} \left| \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iN\gamma(t)} f(x-t)\,\frac{\mathrm{d}t}{t} \right|, \quad \forall\, x \in \mathbb{R},$$

for any $f \in \mathcal{S}(\mathbb{R})$. Lie [25] considered the quadratic *Carleson operator* $\mathcal{C}_2$, which is defined by

$$\mathcal{C}_2 f(x) := \sup_{N_1,N_2\in\mathbb{R}} \left| \mathrm{p.\,v.} \int_{\mathbb{T}} e^{iN_1 t + iN_2 t^2} f(x-t)\,\frac{\mathrm{d}t}{t} \right|, \quad \forall\, x \in \mathbb{T},$$

for any $f \in C^1(\mathbb{T})$, where $\mathbb{T} := [-\frac{1}{2}, \frac{1}{2}]$. He obtained its $L^p(\mathbb{T})$ boundedness for any given $p \in [1, 2)$ and weak-$(2, 2)$ boundedness. Later, Guo [16, Corollary 1.7] obtained the $L^2(\mathbb{R})$ boundedness of the *Carleson operator* $\mathcal{C}_\alpha$ defined as

$$\mathcal{C}_\alpha f(x) := \sup_{N_1,N_2\in\mathbb{R}} \left| \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iN_1 t + iN_2 [t]^\alpha} f(x-t)\,\frac{\mathrm{d}t}{t} \right|, \quad \forall\, x \in \mathbb{R},$$

for any $f \in \mathcal{S}(\mathbb{R})$, where $[t]^\alpha$ stands for either $|t|^\alpha$ or $\operatorname{sgn}(t)|t|^\alpha$ for any $t \in \mathbb{R}$, $\alpha \in (0, \infty)$, $\alpha \neq 1$, $\alpha \neq 2$. More recently, the first author [33] obtained the $L^p(\mathbb{R})$ boundedness of the *Carleson operator* $\mathcal{C}_{\alpha,\beta}$, which is defined by setting, for any $f \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{C}_{\alpha,\beta} f(x) := \sup_{N_1,N_2\in\mathbb{R}} \left| \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iN_1 [t]^\alpha + iN_2 [t]^\beta} f(x-t)\,\frac{\mathrm{d}t}{t} \right|, \quad \forall\, x \in \mathbb{R},$$

where $[t]^\alpha$ stands for either $|t|^\alpha$ or $\operatorname{sgn}(t)|t|^\alpha$, $[t]^\beta$ stands for either $|t|^\beta$ or $\operatorname{sgn}(t)|t|^\beta$ and $\alpha, \beta, p \in (1, \infty)$. Based on the Carleson operators $\mathcal{C}_{N,\gamma}$, $\mathcal{C}_\alpha$ and $\mathcal{C}_{\alpha,\beta}$, it is natural to consider the $L^p(\mathbb{R})$ boundedness of the Carleson operator

$$\mathcal{C}_{N_1,N_2,\gamma}f(x) := \sup_{N_1,N_2\in\mathbb{R}} \left| \text{p. v.} \int_{-\infty}^{\infty} e^{iN_1t+iN_2\gamma(t)} f(x-t)\,\frac{\mathrm{d}t}{t} \right|, \quad \forall x \in \mathbb{R},$$

original defined for any $f \in \mathcal{S}(\mathbb{R})$, where $p \in (1,\infty)$. By linearizing the supremum, it suffices to consider the Carleson operator $\mathcal{C}_{u_1,u_2,\gamma}$ in (1.3), we obtain

Theorem 1.7. *Let u_1, u_2 and γ be the same as in Theorem 1.1, and $p \in (1,\infty)$. Then there exists a positive constant C such that*

$$\|\mathcal{C}_{u_1,u_2,\gamma}f\|_{L^p(\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R})}$$

for any $f \in L^p(\mathbb{R})$ with the bound C independent of u_1 and u_2.

From Remark 1.6(i), we do not establish the $L^2(\mathbb{R})$ boundedness of Carleson operator $\mathcal{C}_{N_1,N_2,\gamma}$ for $\gamma(t) := t^2$, but for all $\gamma(t) := [t]^\alpha, \alpha \in (1,\infty), \alpha \neq 2$. The case $\gamma(t) := t^2$ is fundamentally different due to the presence of the symmetry of quadratic modulations. Therefore, the weak-$(2,2)$ boundedness of $\mathcal{C}_2$ in a sense should be regarded as an endpoint estimate for our results of Theorem 1.7 and Guo [16, Corollary 1.7].

The layout of the paper is as follows. In Section 2, we give a detailed proof of Theorem 1.7. It then gives the $L^2(\mathbb{R}^2)$ boundedness of (1.1), that is Theorem 1.1. Section 3 is devoted to obtaining the single annulus $L^p(\mathbb{R}^2)$ estimate for (1.1) for any given $p \in (1,\infty)$, it is Theorem 1.2.

2. Boundedness of $H_{u_1,u_2,\gamma}$ and $\mathcal{C}_{u_1,u_2,\gamma}$

In this section, we prove Theorems 1.1 and 1.7. We start with some standard reductions that Theorem 1.1 follows from establishing Theorem 1.7. As in [26], we obtain

$$\|H_{u_1,u_2,\gamma}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \le \sup_{\lambda\in\mathbb{R}} \|S_\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}$$

by Plancherel's formula, where

$$S_\lambda f(x) := \text{p. v.} \int_{-\infty}^{\infty} e^{-i\lambda u_1(x)t-i\lambda u_2(x)\gamma(t)} f(x-t)\,\frac{\mathrm{d}t}{t}, \quad \forall x \in \mathbb{R}.$$

Since we pursue the Hilbert transform $H_{u_1,u_2,\gamma}$ is bounded on $L^2(\mathbb{R}^2)$ with a bound that can be taken to be independent of the measurable functions u_1 and u_2, we need only to obtain the $L^2(\mathbb{R})$ boundedness of

$$\mathcal{C}_{u_1,u_2,\gamma}f(x) = \text{p. v.} \int_{-\infty}^{\infty} e^{iu_1(x)t+iu_2(x)\gamma(t)} f(x-t)\,\frac{\mathrm{d}t}{t}, \quad \forall x \in \mathbb{R},$$

with a bound independent of the measurable functions u_1 and u_2. Therefore, it is enough to prove Theorem 1.7. We prove it by establishing the following four steps:

2.1. Basic dyadic decomposition

The first step is to break up $\mathcal{C}_{u_1,u_2,\gamma}$ into pieces which are supported on dyadic annuli about the integral variable. Recall that the non-negative smooth bump function $\psi: \mathbb{R} \to \mathbb{R}$ is supported on $\{t \in \mathbb{R} : \frac{1}{2} \le |t| \le 2\}$ satisfying $\Sigma_{l\in\mathbb{Z}}\psi_l(t) = 1$ for any $t \neq 0$, where $\psi_l(t) = \psi(2^{-l}t)$. On the other hand, since $\gamma(0) = \gamma'(0) = 0$,

$\gamma \in C^4(0,\infty)$ is convex on $(0,\infty)$, it follows that $\gamma'' \geq 0$ on $(0,\infty)$, then γ' is increasing on $(0,\infty)$, which further implies that $\lim_{t\to\infty}\gamma(t) = \infty$. We may define $n:\ \mathbb{R} \to \mathbb{Z}$ such that

$$\frac{1}{\gamma(2^{n(x)+1})} \leq |u_2(x)| \leq \frac{1}{\gamma(2^{n(x)})}, \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Let us set

$$\mathcal{C}_{u_1,u_2,\gamma,k}f(x) := \int_{-\infty}^{\infty} e^{iu_1(x)t+iu_2(x)\gamma(t)} f(x-t)\psi_k(t)\,\frac{\mathrm{d}t}{t}, \quad \forall x \in \mathbb{R}.$$

We decompose the Carleson operator $\mathcal{C}_{u_1,u_2,\gamma}$ into the following low frequency part $\mathcal{C}^{(1)}_{u_1,u_2,\gamma}$ and high frequency part $\mathcal{C}^{(2)}_{u_1,u_2,\gamma}$:

$$\begin{aligned}\mathcal{C}_{u_1,u_2,\gamma}f(x) &= \sum_{k\leq n(x)-1} \mathcal{C}_{u_1,u_2,\gamma,k}f(x) + \sum_{k\geq n(x)} \mathcal{C}_{u_1,u_2,\gamma,k}f(x) \\ &=: \mathcal{C}^{(1)}_{u_1,u_2,\gamma}f(x) + \mathcal{C}^{(2)}_{u_1,u_2,\gamma}f(x).\end{aligned} \tag{2.2}$$

2.2. Low frequency part $\mathcal{C}^{(1)}_{u_1,u_2,\gamma}$

The low frequency part $\mathcal{C}^{(1)}_{u_1,u_2,\gamma}$ is further divided into the following two parts:

$$\mathcal{C}^{(1)}_{u_1,u_2,\gamma}f(x) = T_1f(x) + T_2f(x).$$

The former part T_1f is defined as

$$T_1f(x) := \mathrm{p.\,v.} \int_{|t|\leq 2^{n(x)}} \left[e^{iu_1(x)t+iu_2(x)\gamma(t)} - e^{iu_1(x)t}\right] f(x-t)\phi(t)\,\frac{\mathrm{d}t}{t}$$

and the latter part T_2f is defined as

$$T_2f(x) := \mathrm{p.\,v.} \int_{|t|\leq 2^{n(x)}} e^{iu_1(x)t} f(x-t)\phi(t)\,\frac{\mathrm{d}t}{t},$$

where $\phi(t) := \sum_{k\leq n(x)-1}\psi_k(t)$.

For the former part T_1f, noticing the fact that γ' is increasing on $(0,\infty)$ and $\gamma(0) = 0$, by a simple calculation, this implies that $\frac{\gamma(t)}{t}$ is increasing on $(0,\infty)$. Combining γ is either odd or even and (2.1), we get that

$$\begin{aligned}T_1f(x) &\leq \int_{|t|\leq 2^{n(x)}} |f(x-t)|\,|u_2(x)|\,\frac{\gamma(2^{n(x)})}{2^{n(x)}}\phi(t)\,\mathrm{d}t \\ &\leq \frac{1}{2^{n(x)}} \int_{|t|\leq 2^{n(x)}} |f(x-t)|\,\mathrm{d}t \lesssim Mf(x).\end{aligned}$$

Here and hereafter, M denotes the Hardy-Littlewood maximal operator.

For the latter part $T_2 f$, by linearizing the supremum, we deduce that

$$\begin{aligned}|T_2 f(x)| &= \left| \int\limits_{|t|\le 2^{n(x)}} e^{iu_1(x)t} f(x-t)\frac{\phi(t)-1}{t}\,\mathrm{d}t + \text{p. v.} \int\limits_{|t|\le 2^{n(x)}} e^{iu_1(x)t} f(x-t)\,\frac{\mathrm{d}t}{t}\right| \\ &\le \int\limits_{2^{n(x)-1}\le |t|\le 2^{n(x)}} |f(x-t)| \left|\frac{\phi(t)-1}{t}\right| \mathrm{d}t + \mathcal{C}^* f(x) \\ &\le \frac{1}{2^{n(x)-1}} \int\limits_{|t|\le 2^{n(x)}} |f(x-t)|\,\mathrm{d}t + \mathcal{C}^* f(x) \lesssim Mf(x) + \mathcal{C}^* f(x).\end{aligned}$$

Here $\mathcal{C}^*$ is the *maximal truncated Carleson operator* defined as

$$\mathcal{C}^* f(x) := \sup_{N\in\mathbb{R},\varepsilon>0} \left| \text{p. v.} \int\limits_{|t|<\varepsilon} e^{iNt} f(x-t)\,\frac{\mathrm{d}t}{t}\right|, \quad \forall\, x\in\mathbb{R}.$$

From [12, Lemma 6.3.2], we have that

$$\mathcal{C}^* f(x) \lesssim Mf(x) + M(\mathcal{C}f)(x),$$

where $\mathcal{C}$ is the Carleson operator in (1.11). Therefore,

$$\mathcal{C}^{(1)}_{u_1,u_2,\gamma} f(x) \lesssim Mf(x) + M(\mathcal{C}f)(x).$$

Since both M and $\mathcal{C}$ are known to be bounded on $L^p(\mathbb{R})$, we may conclude that

$$\left\|\mathcal{C}^{(1)}_{u_1,u_2,\gamma} f\right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}$$

as desired, where $p\in(1,\infty)$.

2.3. High frequency part $\mathcal{C}^{(2)}_{u_1,u_2,\gamma}$

For the high frequency part $\mathcal{C}^{(2)}_{u_1,u_2,\gamma}$, we write it as a series of operators $\{S_k\}_{k=0}^{\infty}$. We want to get a decay estimate for each of S_k. We shall adapt the TT^* argument from Stein and Wainger in [30]. Indeed,

$$\mathcal{C}^{(2)}_{u_1,u_2,\gamma} f(x) = \sum_{k\ge 0} \int_{-\infty}^{\infty} e^{iu_1(x)t+iu_2(x)\gamma(t)} f(x-t)\psi_{k+n(x)}(t)\,\frac{\mathrm{d}t}{t} =: \sum_{k\ge 0} S_k f(x).$$

For each fixed $k\ge 0$, we trivially have that

$$\begin{aligned}|S_k f(x)| &\le \int\limits_{2^{k+n(x)-1}\le|t|\le 2^{k+n(x)+1}} |f(x-t)|\,\frac{|\psi_{k+n(x)}(t)|}{|t|}\,\mathrm{d}t \\ &\le \frac{1}{2^{k+n(x)-1}} \int\limits_{|t|\le 2^{k+n(x)+1}} |f(x-t)|\,\mathrm{d}t \lesssim Mf(x).\end{aligned}$$

From this and the well-known boundedness of M on $L^p(\mathbb{R})$ for any given $p \in (1,\infty)$, it follows that $\|S_k f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}$, by interpolation, the proof of $\|\mathcal{C}^{(2)}_{u_1,u_2,\gamma} f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}$ for any given $p \in (1,\infty)$ would be completed if we could show that there exists a positive constant ω_0 such that, for any $k \geq 0$,

$$\|S_k f\|_{L^2(\mathbb{R})} \lesssim 2^{-\omega_0 k} \|f\|_{L^2(\mathbb{R})}. \tag{2.3}$$

Based on the standard TT^* argument, we first write the operator $S_k {S_k}^*$ as

$$\begin{aligned}\mathrm{p.\,v.} \int_{-\infty}^{\infty} \mathrm{p.\,v.} \int_{-\infty}^{\infty} \; & e^{-iu_1(z)(z-x+t)-iu_2(z)\gamma(z-x+t)} \frac{\psi(2^{-n(z)-k}(z-x+t))}{z-x+t} \\ & \times \; e^{iu_1(x)t+iu_2(x)\gamma(t)} \frac{\psi(2^{-n(x)-k}t)}{t} \,\mathrm{d}t f(z)\,\mathrm{d}z.\end{aligned}$$

For any $t \in \mathbb{R}$, let

$$\Phi_k^{u_1(x),u_2(x)}(t) := e^{iu_1(x)t+iu_2(x)\gamma(t)} \frac{\psi(2^{-n(x)-k}t)}{t}$$

and

$$\tilde{\Phi}_k^{u_1(x),u_2(x)}(t) := \bar{\Phi}_k^{u_1(x),u_2(x)}(-t).$$

Therefore, we immediately get

$$S_k {S_k}^* f(x) = \mathrm{p.\,v.} \int_{-\infty}^{\infty} \tilde{\Phi}_k^{u_1(z),u_2(z)} * \Phi_k^{u_1(x),u_2(x)}(x-z) f(z)\,\mathrm{d}z.$$

From now on, we may assume for simplicity that $2^{n(x)} \leq 2^{n(z)}$. Let $\xi := x - z$, we have that the kernel of $S_k {S_k}^*$ is given by

$$\mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{-iu_1(z)(-\xi+t)-iu_2(z)\gamma(-\xi+t)} \frac{\psi(2^{-n(z)-k}(-\xi+t))}{-\xi+t} e^{iu_1(x)t+iu_2(x)\gamma(t)} \frac{\psi(2^{-n(x)-k}t)}{t}\,\mathrm{d}t. \tag{2.4}$$

Set $\eta := 2^{-n(x)-k}t$ and hence (2.4) has the form

$$\begin{aligned}\mathrm{p.\,v.} \int_{-\infty}^{\infty} \; & e^{-iu_1(z)(-\xi+2^{n(x)+k}\eta)-iu_2(z)\gamma(-\xi+2^{n(x)+k}\eta)} \frac{\psi(-\xi 2^{-n(z)-k} + \frac{2^{n(x)}}{2^{n(z)}}\eta)}{-\xi + 2^{n(x)+k}\eta} \\ & e^{iu_1(x)2^{n(x)+k}\eta+iu_2(x)\gamma(2^{n(x)+k}\eta)} \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta.\end{aligned} \tag{2.5}$$

Furthermore, let us set $0 < h := \frac{2^{n(x)}}{2^{n(z)}} \leq 1$, $s := \frac{\xi}{2^{n(z)+k}}$ and $t := \eta$, we can rewrite the kernel as

$$\frac{1}{2^{n(z)+k}} \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iu_1(x)2^{n(x)+k}t+iu_2(x)\gamma(2^{n(x)+k}t)-iu_1(z)(2^{n(z)+k}[ht-s])-iu_2(z)\gamma(2^{n(z)+k}[ht-s])} \frac{\psi\,(ht-s)}{ht-s} \frac{\psi(t)}{t}\,\mathrm{d}t. \tag{2.6}$$

To bound the above integral, we shall need to use a crucial estimate from the following Proposition 2.1 that will allow us to control $S_k{S_k}^*$ by Hardy-Littlewood maximal operator M with bound 2^{-kr_0}. This is the desired estimate for $S_k{S_k}^*$. Let us postpone the proof of Proposition 2.1 for the moment. We now turn to $S_k{S_k}^*$, notice that $\frac{x-z}{2^{n(z)+k}} = s$, by (2.7) of Proposition 2.1, we have that

$$
\begin{aligned}
|S_k{S_k}^*f(x)| &= \Bigg| \int_{-\infty}^{\infty} \frac{1}{2^{n(z)+k}} \,\mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iu_1(x)2^{n(x)+k}t+iu_2(x)\gamma(2^{n(x)+k}t)-iu_1(z)(2^{n(z)+k}[ht-s])-iu_2(z)\gamma(2^{n(z)+k}[ht-s])} \\
&\quad \times \frac{\psi\,(ht-s)}{ht-s}\frac{\psi(t)}{t}\,\mathrm{d}t f(z)\,\mathrm{d}z \Bigg| \\
&\lesssim \int_{-\infty}^{\infty} \frac{1}{2^{n(z)+k}} \left\{ \chi_{[-2^{-kr_1},2^{-kr_1}]}(s) + 2^{-kr_2}\chi_{[-4,4]}(s) \right\} |f(z)|\,\mathrm{d}z \\
&\lesssim \frac{2^{-kr_1}}{2^{n(z)+k}2^{-kr_1}} \int_{\frac{|x-z|}{2^{n(z)+k}} \le 2^{-kr_1}} |f(z)|\,\mathrm{d}z + \frac{2^{-kr_2}}{2^{n(z)+k}} \int_{\frac{|x-z|}{2^{n(z)+k}} \le 4} |f(z)|\,\mathrm{d}z \\
&\lesssim 2^{-kr_1}Mf(x) + 2^{-kr_2}Mf(x) \lesssim 2^{-kr_0}Mf(x),
\end{aligned}
$$

where $\gamma_0 := \min\{r_1, r_2\}$. From the $L^2(\mathbb{R})$ boundedness of M and hence

$$\|S_k\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|S_k{S_k}^*\|^{\frac{1}{2}}_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \lesssim 2^{-\frac{r_0}{2}k}.$$

Thus, (2.3) holds with $\omega_0 := \frac{r_0}{2}$, which completes the proof of Theorem 1.7.

2.4. A crucial decay estimate

As in [30, Lemma 4.1], we need the following decay estimate (2.7). We will bound the integral in (2.7) by the sum of $\chi_{[-2^{-kr_1},2^{-kr_1}]}(s)$ and $2^{-kr_2}\chi_{[-4,4]}(s)$, not just $2^{-kr_2}\chi_{[-4,4]}(s)$, where $[-2^{-kr_1}, 2^{-kr_1}]$ can be regarded as an exceptional set.

Proposition 2.1. *There exist positive constants r_1 and r_2 such that*

$$
\begin{aligned}
&\left| \mathrm{p.\,v.} \int_{-\infty}^{\infty} e^{iu_1(x)2^{n(x)+k}t+iu_2(x)\gamma(2^{n(x)+k}t)-iu_1(z)(2^{n(z)+k}[ht-s])-iu_2(z)\gamma(2^{n(z)+k}[ht-s])} \frac{\psi\,(ht-s)}{ht-s}\frac{\psi(t)}{t}\,dt \right| \\
&\le C\left\{ \chi_{[-2^{-kr_1},2^{-kr_1}]}(s) + 2^{-kr_2}\chi_{[-4,4]}(s) \right\}
\end{aligned}
\tag{2.7}
$$

for any $k \in \mathbb{N}$ and $x, z, s \in \mathbb{R}$, where C is a positive constant that can be taken to be independent of k, x, z, s, u_1 and u_2.

For the proof of Proposition 2.1, we shall use the following two lemmas:

Lemma 2.2. *[17, Lemma 4.5] Let A be an invertible $n \times n$ matrix and $x \in \mathbb{R}^n$. Then*

$$|Ax| \ge |detA|\|A\|^{1-n}|x|,$$

where $\|A\|$ denotes the matrix norm $\sup_{|x|=1}|Ax|$.

Lemma 2.3. *For any $a, b, c, d \in \mathbb{R}$ and $d > 0$, there are at most a finite number of intervals such that*

$$|a\gamma''(t) - b\gamma''(t-c)| > d \tag{2.8}$$

is established on each of the intervals, and the number of intervals is independent of a, b, c, d, where $t \in \mathbb{R}$.

Proof. We proof of Lemma 2.3 by considering the following five cases:

Case 1 $b = 0$ and $a = 0$. Then (2.8) does not exist. In other words, there is no interval such that (2.8) is established.

Case 2 $b = 0$ and $a \neq 0$. From Theorem 1.1(v), $\gamma'''(t) \neq 0$ on $t \in (0, \infty)$, noticing that $\gamma \in C^4(\mathbb{R})$, then γ'' is strictly monotone on $(0, \infty)$, since γ is either odd or even, then Lemma 2.3 is obtained obviously.

Case 3 $b \neq 0$, $c = 0$ and $a = b$. Then (2.8) does not exist.

Case 4 $b \neq 0$, $c = 0$ and $a \neq b$. Then (2.8) is equivalent to $|(a-b)\gamma''(t)| > d$, as Case 2, we have that Lemma 2.3 is established.

Case 5 $b \neq 0$ and $c \neq 0$. Since (2.8) is equivalent to

$$a\gamma''(t) - b\gamma''(t-c) - d > 0 \quad \text{or} \quad a\gamma''(t) - b\gamma''(t-c) + d < 0.$$

Noticing that $\gamma \in C^4(\mathbb{R})$, it suffices to show that

$$a\gamma'''(t) - b\gamma'''(t-c) = 0 \tag{2.9}$$

has a finite number of solutions including no solution, or there are at most a finite number of intervals such that (2.9) is established on each of the intervals, or both, where the number is independent of a, b, c. From Theorem 1.1(v), $\gamma'''(t) \neq 0$ on $t \in (0, \infty)$, noticing that γ is either odd or even, then $\gamma'''(t) \neq 0$ on $t \in (-\infty, 0) \bigcup (0, \infty)$. It is easy to see that we should only consider $t \neq 0$ and $t \neq c$ for (2.9). Then (2.9) is equivalent to

$$\frac{a}{b} = \frac{\gamma'''(t-c)}{\gamma'''(t)} =: F_c(t), \quad t \neq 0,\ t \neq c,\ t \in \mathbb{R}. \tag{2.10}$$

It is easy to see that, for any $t \neq 0, t \neq c, t \in \mathbb{R}$,

$$F_c'(t) = \frac{\gamma^{(4)}(t-c)\gamma'''(t) - \gamma'''(t-c)\gamma^{(4)}(t)}{(\gamma'''(t))^2} = \frac{\gamma'''(t-c)}{\gamma'''(t)} \left[\frac{\gamma^{(4)}(t-c)}{\gamma'''(t-c)} - \frac{\gamma^{(4)}(t)}{\gamma'''(t)} \right]. \tag{2.11}$$

From Theorem 1.1(v), $\frac{\gamma^{(4)}(t)}{\gamma'''(t)}$ is strictly monotone or equals to a constant on $(0, \infty)$, since γ is either odd or even, then

$$\frac{\gamma^{(4)}(t-c)}{\gamma'''(t-c)} = \frac{\gamma^{(4)}(t)}{\gamma'''(t)}, \quad t \neq 0,\ t \neq c,\ t \in \mathbb{R} \tag{2.12}$$

has a finite number of solutions including no solution, or there are at most a finite number of intervals such that (2.12) is established on each of the intervals, or both, where the number is independent of c. Therefore, $F_c'(t)$ in (2.11) share the same character as (2.12). Then (2.10) also has the same character as (2.12). This finishes the proof of Lemma 2.3. $\square$

Proof of Proposition 2.1. We notice that the non-negative smooth function $\psi:\ \mathbb{R} \to \mathbb{R}$ is supported on $\{t \in \mathbb{R}:\ \frac{1}{2} \leq |t| \leq 2\}$ and $0 < h \leq 1$, thus, $|t| \leq 2$, $|ht - s| \leq 2$ and $|s| \leq 4$. For any given $t \in \mathbb{R}$, let

$$Q(t) := u_1(x)2^{n(x)+k}t + u_2(x)\gamma(2^{n(x)+k}t) - u_1(z)(2^{n(z)+k}[ht-s]) - u_2(z)\gamma(2^{n(z)+k}[ht-s]). \tag{2.13}$$

After a direct computation, we may obtain

$$Q''(t) = u_2(x)2^{2(n(x)+k)}\gamma''(2^{n(x)+k}t) - u_2(z)2^{2(n(z)+k)}\gamma''(2^{n(z)+k}[ht-s])h^2, \tag{2.14}$$

and

$$Q'''(t) = u_2(x)2^{3(n(x)+k)}\gamma'''(2^{n(x)+k}t) - u_2(z)2^{3(n(z)+k)}\gamma'''(2^{n(z)+k}[ht-s])h^3. \tag{2.15}$$

Before getting some estimates on Q'' and Q''', we first obtain some controls about $\frac{t\gamma''(t)}{\gamma'(t)}$ and $\frac{t\gamma'(t)}{\gamma(t)}$. Noticing that $\gamma'(0) = 0$, from the Cauchy mean value theorem, for any given $t \in (0,\infty)$, there exists $t_\xi \in (0,t)$ such that $\frac{t\gamma''(t)}{\gamma'(t)} = \frac{t\gamma''(t)-0\gamma''(0)}{\gamma'(t)-\gamma'(0)} = \frac{\gamma''(t_\xi)+t_\xi\gamma'''(t_\xi)}{\gamma''(t_\xi)}$. Thus, by Theorem 1.1(iii), there exists $C_5 := C_3 + 1$ such that

$$\frac{t\gamma''(t)}{\gamma'(t)} \le C_5 \tag{2.16}$$

for any $t \in (0,\infty)$. For $\frac{t\gamma'(t)}{\gamma(t)}$, noticing that $\gamma(0) = 0$, from the Cauchy mean value theorem, for any given $t \in (0,\infty)$, there exists $t_\zeta \in (0,t)$ such that $\frac{\gamma(t)}{t} = \gamma'(t_\zeta)$. This, combined with the fact that γ' is increasing on $(0,\infty)$ and $\gamma > 0$ on $(0,\infty)$, implies that $\gamma'(t_\zeta) \le \gamma'(t)$ for any $t \in (0,\infty)$. Therefore, $1 \le \frac{t\gamma'(t)}{\gamma(t)}$ for any $t \in (0,\infty)$. Furthermore, from $\gamma(0) = 0$ and the Cauchy mean value theorem, similarly to $\frac{t\gamma''(t)}{\gamma'(t)}$, there exists $C_6 := C_5 + 1$ such that

$$1 \le \frac{t\gamma'(t)}{\gamma(t)} \le C_6 \tag{2.17}$$

for any $t \in (0,\infty)$. Now we come back to the estimates on Q'' and Q''', we need to consider the following two cases:

Case A $0 < h \le \sqrt{\frac{C_2}{16C_1^3C_5C_6}}$.

From (2.1), we can see that

$$\begin{aligned} |Q''(t)| &\ge \left|u_2(x)2^{2(n(x)+k)}\gamma''(2^{n(x)+k}t)\right| - \left|u_2(z)2^{2(n(z)+k)}\gamma''(2^{n(z)+k}[ht-s])\right| h^2 \\ &\ge \left|\frac{1}{\gamma(2^{n(x)+1})}2^{2(n(x)+k)}\gamma''(2^{n(x)+k}t)\right| - \left|\frac{1}{\gamma(2^{n(z)})}2^{2(n(z)+k)}\gamma''(2^{n(z)+k}[ht-s])\right| h^2. \end{aligned} \tag{2.18}$$

By Theorem 1.1(ii) and (2.16), we have that $C_2 \le \frac{t\gamma''(t)}{\gamma'(t)} \le C_5$ for any $t \in (0,\infty)$, noticing γ' is increasing on $(0,\infty)$, $\frac{1}{2} \le |t| \le 2$, $\frac{1}{2} \le |ht-s| \le 2$, we see that the formula in the last term in (2.18) is

$$\begin{aligned} &= \left|\frac{1}{\gamma(2^{n(x)+1})}2^{2(n(x)+k)}\frac{2^{n(x)+k}t\gamma''(2^{n(x)+k}t)}{\gamma'(2^{n(x)+k}t)}\frac{\gamma'(2^{n(x)+k}t)}{2^{n(x)+k}t}\right| \\ &\quad - \left|\frac{1}{\gamma(2^{n(z)})}2^{2(n(z)+k)}\frac{2^{n(z)+k}[ht-s]\gamma''(2^{n(z)+k}[ht-s])}{\gamma'(2^{n(z)+k}[ht-s])}\frac{\gamma'(2^{n(z)+k}[ht-s])}{2^{n(z)+k}[ht-s]}\right| h^2 \\ &\ge \frac{C_2}{2}\left|\frac{1}{\gamma(2^{n(x)+1})}2^{2(n(x)+k)}\frac{\gamma'(2^{n(x)+k}\frac{1}{2})}{2^{n(x)+k}}\right| - 2C_5\left|\frac{1}{\gamma(2^{n(z)})}2^{2(n(z)+k)}\frac{\gamma'(2^{n(z)+k}2)}{2^{n(z)+k}}\right| h^2. \end{aligned} \tag{2.19}$$

Since $\frac{\gamma'(2t)}{\gamma'(t)}$ is decreasing on $(0,\infty)$, then $\frac{\gamma'(2^k t)}{\gamma'(t)} = \frac{\gamma'(2^k t)}{\gamma'(2^{k-1}t)}\frac{\gamma'(2^{k-1}t)}{\gamma'(2^{k-2}t)}\cdots\frac{\gamma'(2t)}{\gamma'(t)}$ is decreasing on $(0,\infty)$ for any $k\in\mathbb{N}$. Noticing (2.17), $\frac{\gamma'(2t)}{\gamma'(t)} \le C_1$ and $0<h\le\sqrt{\frac{C_2}{16C_1^3C_5C_6}}$, combining γ' is increasing on $(0,\infty)$, we obtain that previous display is

$$
\begin{aligned}
=&\frac{C_2}{2}\left|\frac{2^{n(x)+1}\gamma'(2^{n(x)+1})}{\gamma(2^{n(x)+1})}\frac{2^{n(x)+k}}{2^{n(x)+1}}\frac{\gamma'(2^{n(x)+k}\frac{1}{2})}{\gamma'(2^{n(x)+k})}\frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}\frac{\gamma'(2^{n(x)})}{\gamma'(2^{n(x)+1})}\right| \\
&-2C_5\left|\frac{2^{n(z)}\gamma'(2^{n(z)})}{\gamma(2^{n(z)})}\frac{2^{n(z)+k}}{2^{n(z)}}\frac{\gamma'(2^{n(z)+k}2)}{\gamma'(2^{n(x)+k})}\frac{\gamma'(2^{n(z)+k})}{\gamma'(2^{n(z)})}\right|h^2 \\
\ge&\frac{C_2}{4C_1^2}2^k\frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})} - 2C_1C_5C_62^k\frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}h^2 \\
=&\left(\frac{C_2}{4C_1^2}-2C_1C_5C_6h^2\right)2^k\frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})} \ge \left(\frac{C_2}{8C_1^2}\right)2^k.
\end{aligned}
\tag{2.20}
$$

Therefore, $|Q''(t)|\gtrsim 2^k$, by van der Corput's lemma, it is not difficult to prove that the left hand side of (2.7) can be bounded by a uniform constant $2^{-\frac{1}{2}k}$. Then (2.7) is established with $r_2=\frac{1}{2}$ and arbitrary positive constant r_1.

Case B $\sqrt{\frac{C_2}{16C_1^3C_5C_6}}<h\le 1$.

If $|s|\le 2^{-\frac{k}{2}}$, it is easy to see that the integral in (2.7) has a uniform bound C. Thus, in this case, (2.7) holds with $r_1=\frac{1}{2}$ and arbitrary positive constant r_2. We remark that in what follows we only need to consider the case that $|s|\ge 2^{-\frac{k}{2}}$. Let

$$a_0 := \frac{2^{n(x)+k}t\gamma'''(2^{n(x)+k}t)}{\gamma''(2^{n(x)+k}t)}$$

and

$$b_0 := \frac{2^{n(z)+k}(ht-s)\gamma'''(2^{n(z)+k}[ht-s])}{\gamma''(2^{n(z)+k}[ht-s])}.$$

We can write

$$\begin{pmatrix} Q''(t) \\ Q'''(t)\end{pmatrix} = M_{t,s}\cdot\Upsilon, \tag{2.21}$$

where

$$M_{t,s} := \begin{pmatrix} 1 & h^2 \\ a_0\frac{1}{t} & b_0\frac{h^3}{ht-s}\end{pmatrix} \tag{2.22}$$

and

$$\Upsilon := \begin{pmatrix} u_2(x)2^{2(n(x)+k)}\gamma''(2^{n(x)+k}t) \\ -u_2(z)2^{2(n(z)+k)}\gamma''(2^{n(z)+k}[ht-s])\end{pmatrix}. \tag{2.23}$$

By Theorem 1.1(iii) and the fact that γ is either odd or even, and noticing that $\frac{1}{2}\le|t|\le 2$, $\frac{1}{2}\le|ht-s|\le 2$, $\sqrt{\frac{C_2}{16C_1^3C_5C_6}}<h\le 1$, we conclude that

$$|a_0|\le C_3 \quad\text{and}\quad |b_0|\le C_3. \tag{2.24}$$

This further implies that

$$\|M_{t,s}\| = \sup_{|x|=1} |M_{t,s}x| \lesssim 1. \tag{2.25}$$

From (2.22), by Theorem 1.1(iv), together with the fact that $|s| \leq 4$, $h = \frac{2^{n(x)}}{2^{n(z)}}$ and the Cauchy mean value theorem, it follows that there exists a constant $\theta \in [0,1]$ such that

$$\begin{aligned} |\mathrm{det} M_{t,s}| &= h^2 2^{n(x)+k} \left| \frac{\gamma'''(2^{n(x)+k}t - 2^{n(z)+k}s)}{\gamma''(2^{n(x)+k}t - 2^{n(z)+k}s)} - \frac{\gamma'''(2^{n(x)+k}t)}{\gamma''(2^{n(x)+k}t)} \right| \\ &= h^2 2^{n(x)+k} \left| \left(\frac{\gamma'''}{\gamma''} \right)' \left(2^{n(x)+k}t - 2^{n(z)+k}s\theta \right) 2^{n(z)+k}s \right| \\ &\geq C_4 h^2 2^{n(x)+k} \frac{1}{\left[2^{n(x)+k}t - 2^{n(z)+k}s\theta \right]^2} \left| 2^{n(x)+k}s \right| \\ &= C_4 h^2 \frac{1}{\left(t - \frac{1}{h}s\theta \right)^2} |s| \gtrsim 2^{-\frac{k}{2}}. \end{aligned} \tag{2.26}$$

For $|\Upsilon|$, the principle is essentially the same as in the proof of (2.18), we claim that $|\Upsilon|$ satisfies the lower bound

$$\begin{aligned} |\Upsilon| &= \sqrt{\left[u_2(x) 2^{2(n(x)+k)} \gamma''(2^{n(x)+k}t) \right]^2 + \left[-u_2(z) 2^{2(n(z)+k)} \gamma''(2^{n(z)+k}[ht - s]) \right]^2} \\ &\geq \left| u_2(x) 2^{2(n(x)+k)} \gamma''(2^{n(x)+k}t) \right| \geq \left(\frac{C_2}{4C_1^2} \right) 2^k. \end{aligned} \tag{2.27}$$

By Lemma 2.2 with $n = 2$ and a simple computation using (2.25), (2.26), (2.27), we have

$$M_{t,s}\Upsilon \geq |\mathrm{det} M_{t,s}| \|M_{t,s}\|^{-1} |\Upsilon| \gtrsim 2^{\frac{k}{2}}. \tag{2.28}$$

It is then easily verified the pointwise lower bound that

$$\sqrt{[Q''(t)]^2 + [Q'''(t)]^2} \gtrsim 2^{\frac{k}{2}}. \tag{2.29}$$

By pigeonholing, there are now two cases:

Case I $|Q''(t)| \gtrsim 2^{\frac{k}{2}}$.

By Lemma 2.3, noticing that $h = \frac{2^{n(x)}}{2^{n(z)}}$ and letting

$$a := u_2(x) 2^{2(n(x)+k)}, \quad b := u_2(z) 2^{2(n(z)+k)} h^2, \quad c := 2^{n(z)+k}s, \quad d := 2^{\frac{k}{2}} \quad \text{and} \quad t := 2^{n(x)+k}t,$$

we have that this case only happens at most a finite number of intervals, and the number of intervals is independent of x, z, s, k and u. By van der Corput's lemma, we have that the integral in (2.7) on this portion is established with $r_2 = \frac{1}{4}$ and arbitrary positive constant r_1.

Case II $|Q'''(t)| \gtrsim 2^{\frac{k}{2}}$.

As in the treatment of Case I, Case II also only happens at most a finite number of intervals, by van der Corput's lemma, we have that the integral in (2.7) on this portion is established with $r_2 = \frac{1}{6}$ and arbitrary positive constant r_1.

Putting things together, we obtain that the integral in (2.7) is established with $r_2 = \frac{1}{6}$ and arbitrary positive constant r_1 at the Case B. This finishes the proof of Proposition 2.1. □

3. Single annulus $L^p(\mathbb{R}^2)$ estimate for $H_{u_1,u_2,\gamma}$

In this section, we prove Theorem 1.2. There are many other works about single annulus $L^p(\mathbb{R}^2)$ estimate, such as [1] and [20]. The main strategy of our proof is to split our operator into a low frequency part $H^{(1)}_{u_1,u_2,\gamma}P_0 f$ and a high frequency part $H^{(2)}_{u_1,u_2,\gamma}P_0 f$. We want to compare the low frequency part $H^{(1)}_{u_1,u_2,\gamma}P_0 f$ with the maximal truncated Hilbert transform $\tilde{H}P_0 f$. Later, we consider the difference between $H^{(1)}_{u_1,u_2,\gamma}P_0 f$ and $\tilde{H}P_0 f$. For the high frequency part, which is further divided into a series of operators, we want to get a decay estimate for each of these operators.

Proof of Theorem 1.2. By an anisotropic scaling

$$x_1 \to x_1,\ x_2 \to 2^{-l}x_2,$$

we can restrict our proof to $l = 0$. Let

$$H_{u_1,u_2,\gamma,k}P_0 f(x_1, x_2) := \int_{-\infty}^{\infty} P_0 f(x_1 - t, x_2 - u_1(x_1)t - u_2(x_1)\gamma(t))\psi_k(t)\,\frac{\mathrm{d}t}{t}.$$

Let $n :\ \mathbb{R} \to \mathbb{Z}$ be such that for all $x_1 \in \mathbb{R}$,

$$\frac{1}{\gamma(2^{n(x_1)+1})} \le |u_2(x_1)| \le \frac{1}{\gamma(2^{n(x_1)})}. \tag{3.1}$$

We decompose the function $H_{u_1,u_2,\gamma}P_0 f$ into the following two parts:

$$\begin{aligned} H_{u_1,u_2,\gamma}P_0 f(x_1, x_2) &= \sum_{k\le n(x_1)-1} H_{u_1,u_2,\gamma,k}P_0 f(x_1, x_2) + \sum_{k\ge n(x_1)} H_{u_1,u_2,\gamma,k}P_0 f(x_1, x_2) \\ &=: H^{(1)}_{u_1,u_2,\gamma}P_0 f(x_1, x_2) + H^{(2)}_{u_1,u_2,\gamma}P_0 f(x_1, x_2). \end{aligned} \tag{3.2}$$

Let ρ be a non-negative smooth function supported on $\{\xi \in \mathbb{R} :\ \frac{1}{4} \le |\xi| \le 4\}$ and equals to 1 on $\{\xi \in \mathbb{R} :\ \frac{1}{2} \le |\xi| \le 2\}$, furthermore, let us set

$$\mathbb{P}_0 f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - s)\check{\rho}(s)\,\mathrm{d}s.$$

By Fourier transform, after a simple calculation, it is easy to see that

$$\mathbb{P}_0 P_0 f = P_0 f.$$

We first consider $H^{(1)}_{u_1,u_2,\gamma}\mathbb{P}_0 f$. Let $\phi(t) := \sum_{k\le n(x_1)-1} \psi_k(t)$, then

$$H^{(1)}_{u_1,u_2,\gamma}\mathbb{P}_0 f(x_1, x_2) = \mathrm{p.\,v.} \int_{-\infty}^{\infty} \mathbb{P}_0 f(x_1 - t, x_2 - u_1(x_1)t - u_2(x_1)\gamma(t))\phi(t)\,\frac{\mathrm{d}t}{t}. \tag{3.3}$$

For any $(x_1, x_2) \in \mathbb{R}^2$, we define the approximate operator as

$$\tilde{H}\mathbb{P}_0 f(x_1, x_2) := \mathrm{p.\,v.} \int_{-\infty}^{\infty} \mathbb{P}_0 f(x_1 - t, x_2 - u_1(x_1)t)\phi(t)\,\frac{\mathrm{d}t}{t}.$$

We are going to prove the $L^p(\mathbb{R}^2)$ boundedness of $\tilde{H}\mathbb{P}_0$. For the formula [16, (2.6)], letting

$$x := x_1, \quad y := x_2, \quad v(x) := u_1(x_1) \quad \text{and} \quad u(x) := 2^{(-n(x_1)+1)\alpha},$$

by the formula [16, (2.7)], noticing

$$\phi(t) = \sum_{k\le n(x_1)-1} \psi_k(t) = \sum_{k\le 0} \psi_k(2^{-n(x_1)+1}t) =: \phi_0(2^{-n(x_1)+1}t),$$

we have

$$\|\tilde{H}\mathbb{P}_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|\mathbb{P}_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \tag{3.4}$$

for any given $p \in (1,\infty)$ with the bound independent of u_1 and u_2.

Now we look at the difference between $H^{(1)}_{u_1,u_2,\gamma}\mathbb{P}_0 f$ and $\tilde{H}\mathbb{P}_0 f$, which is given by

$$\text{p. v.} \int\limits_{|t|\le 2^{n(x_1)}} \int\limits_{-\infty}^{\infty} f(x_1 - t, x_2 - u_1(x_1)t - z)\,[\check{\rho}(z - u_2(x_1)\gamma(t)) - \check{\rho}(z)]\,\phi(t)\,\mathrm{d}z\,\frac{\mathrm{d}t}{t}. \tag{3.5}$$

Since γ is increasing on $(0,\infty)$ and $|t| \le 2^{n(x_1)}$, we obtain the critical restriction that

$$|u_2(x_1)\gamma(t)| \le |u_2(x_1)|\gamma(2^{n(x_1)}) \le 1.$$

Then we apply the mean value theorem to get that

$$|\check{\rho}(z - u_2(x_1)\gamma(t)) - \check{\rho}(z)| \lesssim \sum_{m\in\mathbb{Z}} \frac{1}{(|m-1|+1)^2} \chi_{[m,m+1]}(z)|u_2(x_1)\gamma(t)|.$$

Since $\sum_{m\in\mathbb{Z}} \frac{1}{(|m-1|+1)^2} \lesssim 1$, it is enough to consider the $L^p(\mathbb{R}^2)$ boundedness of the following operator, for any fixed $m \in \mathbb{Z}$,

$$K_m f(x_1,x_2) := \int\limits_m^{m+1} \text{p. v.} \int\limits_{|t|\le 2^{n(x_1)}} |f(x_1 - t, x_2 - u_1(x_1)t - z)||u_2(x_1)\gamma(t)|\phi(t)\,\frac{\mathrm{d}t}{|t|}\,\mathrm{d}z \tag{3.6}$$

with the bound independent of u_1, u_2 and m. By Minkowski's inequality and the fact that $\frac{\gamma(t)}{t}$ is increasing on $(0,\infty)$, (3.1), we have

$$\begin{aligned}
\|K_m f\|^p_{L^p(\mathbb{R}^2)} &\le \int\limits_{-\infty}^{\infty} \left\{ \int\limits_m^{m+1} \text{p. v.} \int\limits_{|t|\le 2^{n(x_1)}} \|f(x_1 - t, x_2)\|_{L^p(\mathbb{R}^1_{x_2})} \frac{|u_2(x_1)\gamma(t)|}{|t|}\phi(t)\,\mathrm{d}t\,\mathrm{d}z \right\}^p \mathrm{d}x_1 \qquad (3.7) \\
&\le \int\limits_{-\infty}^{\infty} \left\{ \int\limits_{|t|\le 2^{n(x_1)}} \|f(x_1 - t, x_2)\|_{L^p(\mathbb{R}^1_{x_2})} \frac{|u_2(x_1)\gamma(2^{n(x_1)})|}{|2^{n(x_1)}|}\phi(t)\,\mathrm{d}t \right\}^p \mathrm{d}x_1 \\
&\le \int\limits_{-\infty}^{\infty} \left\{ \frac{1}{2^{n(x_1)}} \int\limits_{|t|\le 2^{n(x_1)}} \|f(x_1 - t, x_2)\|_{L^p(\mathbb{R}^1_{x_2})}\,\mathrm{d}t \right\}^p \mathrm{d}x_1
\end{aligned}$$

$$\lesssim \int_{-\infty}^{\infty} \Big[M(\|f(\cdot,x_2)\|_{L^p(\mathbb{R}^1_{x_2})})(x_1)\Big]^p \,\mathrm{d}x_1 \lesssim \|f\|^p_{L^p(\mathbb{R}^2)}.$$

From (3.4) and (3.7), we have

$$\left\|H^{(1)}_{u_1,u_2,\gamma}\mathbb{P}_0 f\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$

Therefore,

$$\left\|H^{(1)}_{u_1,u_2,\gamma}\mathbb{P}_0 P_0 f\right\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)}$$

and hence

$$\left\|H^{(1)}_{u_1,u_2,\gamma} P_0 f\right\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)}\,.$$

This is the desired estimate for the first part $H^{(1)}_{u_1,u_2,\gamma}P_0 f$.

We now turn to the second part $H^{(2)}_{u_1,u_2,\gamma}P_0 f$, which can be written as

$$\begin{aligned} H^{(2)}_{u_1,u_2,\gamma}P_0 f(x_1,x_2) &= \sum_{k\geq 0}\int_{-\infty}^{\infty} P_0 f(x_1-t, x_2-u_1(x_1)t-u_2(x_1)\gamma(t))\psi_{n(x_1)+k}(t)\,\frac{\mathrm{d}t}{t} \\ &=: \sum_{k\geq 0} B_k P_0 f(x_1,x_2). \end{aligned}$$

From the above description, Theorem 1.2 is reduced to the task of obtaining good $L^p(\mathbb{R}^2)$ bounds for each operator B_k. By Minkowski's inequality, we get

$$\begin{aligned} \|B_kP_0f\|^p_{L^p(\mathbb{R}^2)} &\leq \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|P_0 f(x_1-t,x_2-u_1(x_1)t-u_2(x_1)\gamma(t))\frac{\psi_{n(x_1)+k}(t)}{t}\right|^p \mathrm{d}x_2\right]^{\frac{1}{p}}\mathrm{d}t\right\}^p \mathrm{d}x_1 \qquad (3.8) \\ &\leq \int_{-\infty}^{\infty}\left[\int_{2^{n(x_1)+k-1}\leq|t|\leq 2^{n(x_1)+k+1}} \|P_0 f(x_1-t,x_2)\|_{L^p(\mathbb{R}^1_{x_2})}\frac{1}{|t|}\,\mathrm{d}t\right]^p \mathrm{d}x_1 \\ &\lesssim \int_{-\infty}^{\infty}\Big[M(\|P_0f(\cdot,x_2)\|_{L^p(\mathbb{R}^1_{x_2})})(x_1)\Big]^p\,\mathrm{d}x_1 \lesssim \|P_0 f\|^p_{L^p(\mathbb{R}^2)}. \end{aligned}$$

One can see from (2.3) that

$$\left\|\int_{-\infty}^{\infty} e^{iu_1(x)t+iu_2(x)\gamma(t)}P_0 f(x-t)\psi_{k+n(x)}(t)\,\frac{\mathrm{d}t}{t}\right\|_{L^2(\mathbb{R})} \lesssim 2^{-\omega_0 k}\|P_0 f\|_{L^2(\mathbb{R})}.$$

As [26], by the Fourier transform and Plancherel's formula, we assert that

$$\|B_kP_0 f\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-\omega_0 k}\|P_0 f\|_{L^2(\mathbb{R}^2)}. \qquad (3.9)$$

Interpolating between (3.8) and (3.9), sum over $k \geq 0$, we get that

$$\left\| H^{(2)}_{u_1,u_2,\gamma} P_0 f \right\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$. This finishes the proof of Theorem 1.2. $\square$

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